Torsion in magnitude homology of graphs

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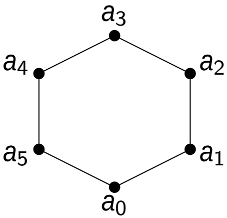
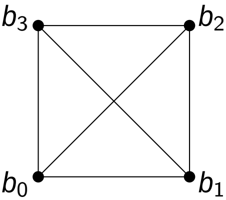
GRAPHS AS METRIC SPACES

Definition

A graph $G = (V; E)$ is a finite collection of vertices and edges (unordered pairs of vertices) with no loops and no multiple edges.





Assigning unit length to each edge, a graph becomes a metric space with $d(x; y)$ = length of a shortest path joining x and y.

E.g. $d(a_1; a_5) = 2$, $d(a_0; a_3) = 3$ while $d(b_i; b_j) = {}_{ij}$.

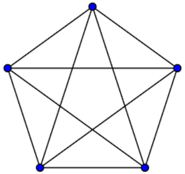
Definition (Leinster)

Let G be a graph with (ordered) vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The magnitude $\#G = \#(q)$ of a graph G is the sum of the entries of the matrix $Z_G^{-1}(q)$, where

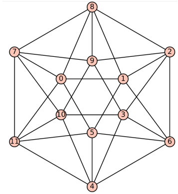
$$Z_G(q) = \begin{bmatrix} q^{d(v_1,v_1)} & q^{d(v_1,v_2)} & \ldots & q^{d(v_1,v_n)} \\ q^{d(v_2,v_1)} & \ddots & & q^{d(v_2,v_n)} \\ \vdots & & & \vdots \\ q^{d(v_n,v_1)} & q^{d(v_n,v_2)} & \ldots & q^{d(v_n,v_n)} \end{bmatrix}$$

Observation (Leinster): $Z_G(0) = I$, so $\det(Z_G(q))$ is invertible in $\mathbb{Z}[[q]]$. Hence $\#(G)(q)$ is a power series in q with integral coefficients.

MAGNITUDE OF COMPLETE AND ICOSAHEDRAL GRAPHS



$$\#K_n(q) = \sum_{\ell=0}^{\infty} n(1-n)^\ell q^\ell$$



$$\#G(q) = 12 - 60q + 240q^2 - 912q^3 +$$

Magnitude is a generalization of cardinality of sets to graphs:

Theorem (Leinster)

Let G and H be any graphs. Then, $\#(G\,2H) = \#(G)\quad \#(H)$

Theorem (Leinster)

Let $(G; H_1; H_2)$ be a projecting decomposition of a graph G. Then,
$\#(G) = \#(H_1) + \#(H_2) \quad \#(H_1 \setminus H_2)$.

You might be familier with the fact that alternating sum formulas are often a good starting point for categorification: it is an approach used to lift Jones (Kauffman bracket) polynomial to Khovanov homology.

Theorem (Leinster)

Let G be a graph. Then, the coefficient of q^ℓ in $\#G(q)$ is given by

$$[\#G(q)]_{q^\ell} = \sum_{k=0}^{\infty} (-1)^k \left|\{x = (x_0, x_1, \dots, x_k) \mid x_i \neq x_{i+1}, \ell(x) = \ell\}\right|.$$

For each $\ell \geq 0$ define a chain complex $(MC_{*,\ell}(G))$ as follows:

Chain groups: $MC_{k,\ell}(G) = \mathbb{Z}\langle\, x = (x_0, x_1, \dots, x_k) \mid \ell(x) = \ell \,\rangle$

Differential: $\partial : MC_{k,\ell}(G) \to MC_{k-1,\ell}(G)$ $\quad \partial = \sum_{i=1}^{k-1} (-1)^i \partial_i$

$$\partial_i(x_0, x_1, \dots, x_k) = \begin{cases} (x_0, \dots, \hat{x}_i, \dots, x_k) & \text{if } \ell(x_0, \dots, \hat{x}_i, \dots, x_k) = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Hepworth, Willerton)

The magnitude homology of a graph G is the bigraded abelian group $\mathrm{MH}(G)$ given in bigrading (k,ℓ) by $\mathrm{MH}_{k,\ell}(G) = H_k(\mathrm{MC}_{*,\ell})$. Magnitude homology is an invariant of graphs with graded Euler characteristic the magnitude power series.

$$\chi_q(\mathrm{MC}_{*,*}(G)) = \sum_{\ell \geq 0} \left(\sum_{k \geq 0} (-1)^k \operatorname{rank}(\mathrm{MH}_{k,\ell}(G)) \right) q^\ell$$

$$= \sum_{\ell \geq 0} \left(\sum_{k \geq 0} (-1)^k \operatorname{rank}(\mathrm{MC}_{k,\ell}(G)) \right) q^\ell$$

$$= \#G(q).$$

LIFTING PROPERTIES OF MAGNITUDE TO MAGNITUDE HOMOLOGY

Theorem (Hepworth, Willerton)

Let G and H be any graphs. Then, magnitude homology groups satisfy a split exact sequence

$$0 \; ! \; MH_{\;;}(G) \quad MH_{\;;}(H) \; ! \; MH_{\;;}(G\,2\,H) \; !$$

$$! \; \mathrm{Tor}_1^{Z}(MH_{\;+1;}(G); MH_{\;;}(H)) \; ! \; 0$$

Theorem (Hepworth, Willerton)

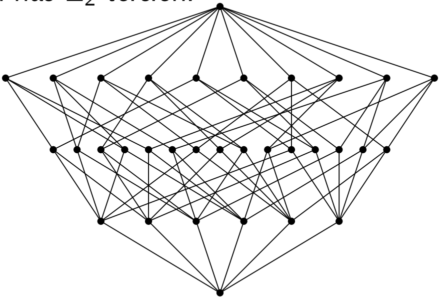
Let $(G; H_1; H_2)$ be a projecting decomposition of a graph G. Then, magnitude homology groups satisfy a split exact sequence

$$0 \; ! \; MH_{\;;}(H_1 \setminus H_2) \; ! \; MH_{\;;}(H_1) \quad MH_{\;;}(H_2) \; ! \; MH_{\;;}(G) \; ! \; 0:$$

Hundreds of computations in Sage, inspired Hepworth and Willerton to conjecture magnitude homology to be torsion free for every graph.

Theorem (Kaneta, Yohsinaga)

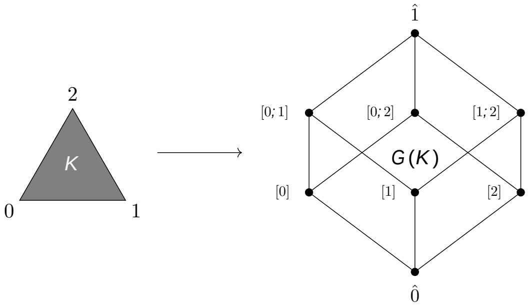
The following graph has Z_2 torsion:



Definition (Kaneta-Yoshinaga graphs)

Let K be a simplicial complex, $P(K)$ be the face poset of K, and let $\hat{P}(K)$ be the poset obtained by augmenting $P(K)$ with unique minimal and maximal elements $\hat{0}$ and $\hat{1}$, respectively. The Kaneta-Yoshinaga graph associated to K is the graph $G(K)$ obtained as the underlying graph of the Hasse diagram of $\hat{P}(K)$.

E.g.



Theorem (Kaneta, Yohsinaga)

Let K be a triangulation of a manifold M and let $\ell = d(\hat{0}, \hat{1})$ in $\hat{P}(K)$. For each $k \geq 1$ there is an embedding,

$$\tilde{H}_{k-2}(M) \hookrightarrow \mathrm{MH}_{k,\ell}(G(K)).$$

Observation

Let K be a triangulation of an m-dimensional manifold M. If K contains a single m-simplex, we obtain embeddings

$$\tilde{H}_{k-2}(M) \hookrightarrow \mathrm{MH}_{k,m+1}(G(K))$$

Otherwise, we obtain embeddings $\tilde{H}_{k-2}(M) \hookrightarrow \mathrm{MH}_{k,m+2}(G(K))$.

$\mathbb{Z}_2$ torsion on the second diagonal of magnitude homology

Theorem (S.-Summers)

For any odd integer $k \geq 3$, there is a graph G such that $MH_{k,k+1}(G)$ contains a subgroup isomorphic to $\mathbb{Z}_2$.

Proof idea:

- Take K to be a triangulation of $\mathbb{RP}^{k-1}$
- Use the embedding

$$\widetilde{H}_{k-2}(\mathbb{RP}^{k-1}) \hookrightarrow MH_{k,k+1}(G(K))$$

described on the previous slide.

Definition (Generalized lens space)

Let $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} |z_i|^2 = 1\}$ be the unit sphere in $\mathbb{C}^{n+1}$. Let $p, q_1, q_2, \ldots, q_n$ be integers with $\gcd(p, q_i) = 1$ for each $1 \le i \le n$. Consider the action of $\mathbb{Z}_p$ on S^{2n+1} defined for each $g \in \mathbb{Z}_p$ by

$$g \cdot (z_0, z_1, \ldots, z_n) = (z_0 e^{\frac{2\pi i g}{p}}, z_1 e^{\frac{2\pi i g q_1}{p}}, z_2 e^{\frac{2\pi i g q_2}{p}}, \ldots, z_n e^{\frac{2\pi i g q_n}{p}}).$$

The lens space $L(p; q_1, q_2, \ldots, q_n)$ is the quotient space $S^{2n+1}/\mathbb{Z}_p$.

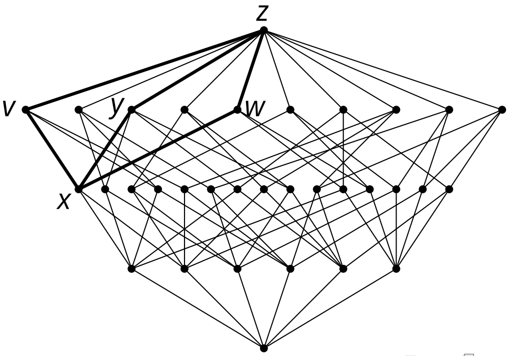
Theorem (S.-Summers)

For each prime p and positive integer r, there is a graph G with $\mathbb{Z}_{p^r}$ torsion in $\mathrm{MH}(G)$. More specifically, for integers $n, r \ge 1$ and each prime p, there is a graph G such that $\mathrm{MH}_{3,2n+3}(G)$ contains $\mathbb{Z}_{p^r}$ torsion.

Proof idea: $L(p^r; q_1, q_2, \ldots, q_n)$ is a triangulable $(2n+1)$-dimensional manifold with fundamental group isomorphic to $\mathbb{Z}_{p^r}$.

Theorem (S.-Summers)

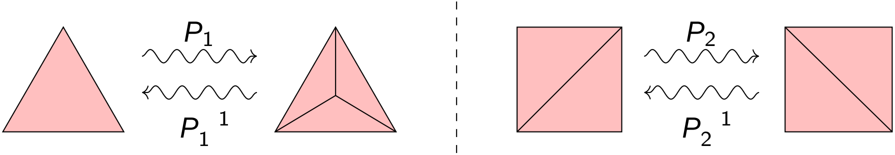
There is a graph G, not obtained from a triangulation via the Kaneta-Yoshinaga construction, with torsion of order two in magnitude homology:



Definition (Pachner)

Let K be a triangulation of an m-manifold M. Let $A \subset K$ be a subcomplex of dimension m, and let $\varphi : A \to A' \subset \partial\Delta^{m+1}$ be a simplicial isomorphism. The Pachner move associated to the triple (K, A, φ) is the adjunction space

$$P_\varphi K := (K - A) \sqcup_\varphi (\partial\Delta^{m+1} - A').$$



Pachner moves on simplicial complexes of dimension 2.

Theorem (S.-Summers)

Let K and K' be triangulations of a manifold M related by a finite sequence of Pachner moves. For each $k \geq 1$, both $\mathrm{MH}_{k,\ell}(G(K))$ and $\mathrm{MH}_{k,\ell}(G(K'))$ have a subgroup isomorphic to $\tilde{H}_{k-2}(M)$.

Theorem

Let $k \geq 3$ be an integer. There exist infinitely many distinct classes of graphs whose magnitude homology contains $\mathbb{Z}_2$ torsion in bigrading $(k, k+1)$.

Proof idea: Form a nested sequence of triangulations of $\mathbb{RP}^{k-1}$ by repeated application of Pachner moves, then appeal to the Kaneta-Yoshinaga embedding.

FAMILIES OF GRAPHS WITH TORSION IN MAGNITUDE HOMOLOGY

Theorem (S.-Summers)

Let p be a prime and $n, m \geq 1$ integers. There exist infinitely many distinct isomorphism classes of graphs whose magnitude homology contains $\mathbb{Z}_{p^r}$ torsion in bigrading $(3, 2n+3)$.

Proof idea: Form a nested sequence of triangulations of generalized lens spaces $L(p^r; q_1, q_2, \ldots, q_n)$ by repeated application of Pachner moves, then appeal to the Kaneta-Yoshinaga embedding.

Theorem (S.-Summers)

Let M be any finitely generated finite abelian group. Then, there exists a graph G whose magnitude homology $\mathrm{MH}(G)$ *contains a subgroup isomorphic to M.*

Proof idea: By the fundamental theorem of finitely generated abelian groups, $M = Z^r \quad Z_{p_1^{r_1}} \quad Z_{p_2^{r_2}} \qquad Z_{p_m^{r_m}}$. The homology groups of a connected sum of manifolds is isomorphic to the direct sum of their respective homology groups. Therefore,

$$H_1(L(p^{r_1};1)\ \#L(p^{r_2};1)\ \# \qquad \#\ L(p^{r_m};1)) = Z_{p_1^{r_1}} \quad Z_{p_2^{r_2}} \qquad Z_{p_m^{r_m}}:$$

Let K be a triangulation of $L(p^{r_1};1)\ \#L(p^{r_2};1)\ \# \qquad \#\ L(p^{r_m};1)$, then appeal to the Kaneta-Yoshinaga embedding.

Theorem (S.-Summers)

Let G be a graph with vertex set V and edge set E. If G has no 3- or 4-cycles, then the first diagonal in the magnitude homology of G satisfies

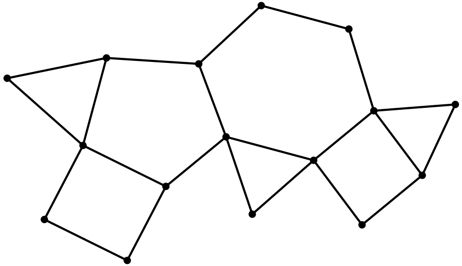
$$\mathrm{MH}_{k,k}(G) = \begin{cases} \mathbb{Z}^{|V|} & k = 0, \\ \mathbb{Z}^{2|E|} & k > 0. \end{cases}$$

Proof idea: Show $B = \{(v,w,v,\ldots,w), (w,v,w,\ldots,v) \mid \{v,w\} \in E\}$ is a basis for the kernel of $\partial : \mathrm{MC}_{k,k}(G) \to \mathrm{MC}_{k-1,k}(G)$: clearly $B \subseteq \ker(\partial)$, and no linear combinations of other generators of $\mathrm{MC}_{k,k}$ can lie in the kernel, else G has a C_3 or C_4 as a subgraph.

Definition

A graph G is said to be outerplanar if G is planar and has a planar drawing in which each vertex lies on an outer face of G.

Such graphs can be constructed from a collection of cycle graphs by gluing along single edges or vertices.



MAGNITUDE HOMOLOGY GROUPS OF EVEN CYCLE GRAPHS

Theorem (Gu)

Fix an integer $m \geq 3$. The magnitude homology of the cycle graph C_{2m} is described as follows.

(1) *All groups $\mathrm{MH}_{k,\ell}(C_{2m})$ are torsion-free.*

(2) *Define a function $T : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as*

1. $T(k,\ell) = 0$ *if* $k < 0$ *or* $\ell < 0$;
2. $T(0,0) = 2m$; $T(1,1) = 4m$;
3. $T(k,\ell) = \max\{T(k-1,\ell-1), T(k-2,\ell-m)\}$ *for* $(k,\ell) \neq (0,0),(1,1)$.

Then, $\mathrm{rank}(\mathrm{MH}_{k,\ell}(C_{2m})) = T(k,\ell)$.

MAGNITUDE HOMOLOGY GROUPS OF EVEN CYCLE GRAPHS

	0	1	2	3	4	5	6	7	8	9	10
0	8										
1		16									
2			16								
3				16							
4			8		16						
5				16		16					
6					16		16				
7						16		16			
8					8		16		16		
9						16		16		16	
10							16		16		16

The ranks of the torsion-free magnitude homology groups of the cycle graph C_8 (Hepworth, Willerton, Gu).

Theorem (S.-Summers)

Let G be an outer planar graph with S components C_4, and R component cycles C_{2m}, constructed using edge-gluings only, $m \quad 3$. Let $S^m_{i;j}$ denote the rank of the $MH(G)$ in the j^{th} entry of the i^{th} diagonal, that is, $S^m_{i;j} = \mathrm{rank}(\mathrm{MH}_{2(i \quad 1)+(j \quad 1);m(i \quad 1)+(j \quad 1)}(G))$. Then, $\mathrm{MH}_{k;\ell}(G)$ are all trivial except for the groups on mentioned diagonals, and these satisfy for $i > 1$

$$\mathrm{rank}(S^m_{1;j}) = \begin{matrix} 2mR + 4S \quad 2(R + S \quad 1) & j = 1; \\ 4mR + 4jS \quad 2(R + S \quad 1) & j > 1; \end{matrix}$$

$$\mathrm{rank}(S^m_{i;j}) = \begin{matrix} 2mR + 4S \quad 2(R + S \quad 1) & j = 1; \\ 4mR \quad 2(R + S \quad 1) & j > 1: \end{matrix}$$

FUTURE WORK

Compute magnitude homology groups for all outerplanar graphs

Apply algebraic Morse theory to compute magnitude homology groups for other families of graphs.

Magnitude and magnitude homology are defined for enriched categories; investigate torsion in other settings.

Are there families of links whose Khovanov complex is particularly amenable to the methods of algebraic Morse theory?

Investigate possible relationships between geodesics in a graph and permissible types of torsion.